OXFORD IB DIPLOMA PROGRAMME

PRIOR LEARNING SUPPORT

MATHEMATICS: ANALYSIS AND APPROACHES

HIGHER LEVEL

COURSE COMPANION

ENHANCED ONLINE

Jennifer Chang Wathall
Josip Harcet
Rose Harrison
Lorraine Heinrichs
Marlene Torres-Skoumal

OXFORD

Solving linear equations

'Solve an equation' means 'find the value of the unknown variable' (the letter).

Rearrange the equation so that the unknown variable *x* becomes the subject of the equation. To keep the equation 'balanced' always do the same to both sides.

Example 1

Solve the equation 3x + 5 = 17

Answer

3x + 5 = 17

3x + 5 - 5 = 17 – 5 *subtract 5*

3x = 12

$\frac{3x}{3} = \frac{12}{3}$ *divide by 3*

x = 4

Example 2

Solve the equation 4(x - 5) = 8

Answer

4(x - 5) = 8

$\frac{4(x-5)}{4} = \frac{8}{4}$ *divide by 4*

x - 5 = 2

x - 5 + 5 = 2 + 5 *add 5*

x = 7

Example 3

Solve the equation 7 - 3x = 1

Answer

7 - 3x = 1

7 - 3x - 7 = 1 – 7 *subtract 7*

-3x = -6

$\frac{-3x}{-3} = \frac{-6}{-3}$ *divide by -3*

x = 2

Example 4

Solve the equation 3(2 + 3x) = 5(4 - x)

Answer

3(2 + 3x) = 5(4 - x)

6 + 9x = 20 - 5x

6 + 9x + 5x = 20 - 5x + 5x add 5x

6 + 14x = 20

6 + 14x - 6 = 20 – 6 subtract 6

14x = 14

$\frac{14x}{14} = \frac{14}{14}$ divide by 14

x = 1

Exercise

Solve these equations.

1 3x - 10 = 2

2 $\frac{x}{2} + 5 = 7$

3 5x + 4 = -11

4 3(x + 3) = 18

5 4(2x - 5) = 20

6 $\frac{2}{5}(3x - 7) = 8$

7 21 - 6x = 9

8 12 = 2 – 5x

9 2(11 -3x) = 4

10 4(3 + x) = 3(9 – 2x)

11 2(10 -2x) = 4(3x +1)

12 $\frac{5x + 2}{3} = \frac{3x + 10}{4}$

Answers

1 x = 4	**2** x = 4	**3** x = -3	**4** x = 3
5 x = 5	**6** x = 9	**7** x = -2.5	**8** x = -2
9 x = 3	**10** x = 1.5	**11** x = 1	**12** x = 2

Simplifying expressions involving roots

$\sqrt{2}, 2-\sqrt{3}, 2\sqrt{5}, \frac{\sqrt{3}}{3}$ are irrational numbers that involve square roots. They are called surds.

In calculations, you can use approximate decimals for these types of irrational number, but for more accurate results you can use surds.

Surds are written in their simplest form when:

- there is no surd in the denominator
- the smallest possible whole number is under the √ sign.

If a question asks for an exact value, it means leave your answer in surd form.

Rules of surds

$(\sqrt{a})^2 = a$ $\qquad \sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ $\qquad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Example 1

Simplify

a $\frac{4}{\sqrt{5}}$ $\qquad$ **b** $\frac{3}{\sqrt{3}}$

Answer

a $\frac{4}{\sqrt{5}} = \frac{4}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$ *multiply top and bottom by the same quantity*

$= \frac{4\sqrt{5}}{(\sqrt{5})^2}$

$= \frac{4\sqrt{5}}{5}$

b $\frac{3}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$ *multiply top and bottom by the same quantity*

$= \frac{3\sqrt{3}}{(\sqrt{3})^2}$

$= \frac{3\sqrt{3}}{3}$ *simplify*

$= \sqrt{3}$

Example 2

a $\sqrt{20}$ **b** $\sqrt{8}-\sqrt{18}$

Answer

a $\sqrt{20}=\sqrt{4}\times\sqrt{5}=2\sqrt{5}$ *use* $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$

b $\sqrt{8}-\sqrt{18}=\sqrt{4\times 2}-\sqrt{9\times 2}$ *look for square numbers that divide into 8 and 18*

$=2\sqrt{3}-3\sqrt{2}$ *use these to write 8 and 18 as products*

$=-\sqrt{2}$ *use* $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$

Example 3

Expand the brackets and simplify $(1+\sqrt{2})(1-\sqrt{2})$

Answer

$(1+\sqrt{2})(1-\sqrt{2})=1-\sqrt{2}+\sqrt{2}-(\sqrt{2})^2$ *use (a+b)(c+d)=ac+ad+bc+bd*

$=1-2$

$=-1$

Example 4

Rewrite the fraction $\frac{1}{(1+\sqrt{3})}$ without surds in the denominator.

Answer

$\frac{1}{(1+\sqrt{3})}=\frac{1}{(1+\sqrt{3})}\times\frac{(1-\sqrt{3})}{(1-\sqrt{3})}$ *multiply top and bottom by* $1-\sqrt{3}$

$=\frac{1-\sqrt{3}}{1-3}=\frac{1-\sqrt{3}}{-2}$

Exercise

1 Simplify

a $\frac{1}{\sqrt{2}}$ **b** $\frac{6}{\sqrt{3}}$ **c** $\frac{5}{\sqrt{5}}$ **d** $\frac{10\sqrt{2}}{\sqrt{5}}$ **e** $\sqrt{\frac{2}{5}}$

2 Simplify

a $2\sqrt{12}$ **b** $\sqrt{75}$ **c** $\sqrt{72}$ **d** $3\sqrt{8}$ **e** $5\sqrt{27}$

3 Simplify

a $\sqrt{3}\times\sqrt{12}$ **b** $\sqrt{3}\times\sqrt{27}$ **c** $\sqrt{24}\times\sqrt{32}$ **d** $2\sqrt{3}\times 3\sqrt{2}$ **e** $3\sqrt{5}\times 5\sqrt{75}$

4 Simplify

a $3\sqrt{5}+2\sqrt{5}$ **b** $5\sqrt{2}-3\sqrt{2}$ **c** $2\sqrt{3}+\sqrt{12}$ **d** $\sqrt{2}-\sqrt{8}$ **e** $\sqrt{12}-2\sqrt{3}$

5 Expand and simplify

a $(3+\sqrt{2})^2$ **b** $(\sqrt{2}+\sqrt{3})^2$ **c** $(3+\sqrt{2})(1-\sqrt{2})$ **d** $(2+\sqrt{2})(2-\sqrt{2})$

e $(4+\sqrt{3})(1-\sqrt{2})$

6 Simplify

a $\dfrac{1+\sqrt{3}}{\sqrt{7}}$ **b** $\dfrac{1}{1-2\sqrt{3}}$ **c** $\dfrac{\sqrt{5}}{1+\sqrt{5}}$ **d** $\dfrac{4+\sqrt{2}}{3-2\sqrt{2}}$

7 Write these without a surd on the denominator. Simplify as much as possible.

a $\dfrac{2}{\sqrt{3}}+3\sqrt{3}$ **b** $\dfrac{\sqrt{3}}{2}+\dfrac{5}{\sqrt{3}}$ **c** $\sqrt{20}+\dfrac{2}{\sqrt{5}}$

Answers

1	**a** $\frac{\sqrt{2}}{2}$	**b** $2\sqrt{3}$	**c** $\sqrt{5}$	**d** $2\sqrt{10}$	**e** $\frac{\sqrt{10}}{5}$
2	**a** $4\sqrt{3}$	**b** $5\sqrt{3}$	**c** $6\sqrt{2}$	**d** $6\sqrt{2}$	**e** $15\sqrt{3}$
3	**a** 6	**b** 9	**c** $16\sqrt{3}$	**d** $6\sqrt{6}$	**e** $75\sqrt{15}$
4	**a** $5\sqrt{5}$	**b** $2\sqrt{2}$	**c** $4\sqrt{3}$	**d** $-\sqrt{2}$	**e** 0
5	**a** $11+6\sqrt{2}$	**b** $5+2\sqrt{6}$	**c** $1-2\sqrt{2}$	**d** 2	**e** $4+\sqrt{3}-4\sqrt{2}-\sqrt{6}$
6	**a** $\frac{\sqrt{21}+\sqrt{7}}{7}$	**b** $-\frac{(1+2\sqrt{3})}{11}$	**c** $\frac{5-\sqrt{5}}{4}$	**d** $16+11\sqrt{2}$	
7	**a** $\frac{11\sqrt{3}}{3}$	**b** $\frac{13\sqrt{3}}{6}$	**c** $\frac{12\sqrt{5}}{5}$		

Adding and subtracting algebraic fractions

To add or subtract fractions, first write them over a common denominator.

Example 1

Combine these fractions, simplifying your answer.

a $\frac{x}{2x+1}+\frac{5x+3}{2x+1}$

$\frac{x}{2x+1}+\frac{5x+3}{2x+1}=\frac{x+(5x+3)}{2x+1}$ *keep the common denominator and add the numerators*

$=\frac{6x+3}{2x+1}$ *combine like terms*

$=\frac{3(2x+1)}{2x+1}$ *factorize and simplify whenever possible*

$= 3$

b $\frac{2x-3}{4x-51}-\frac{6x-2}{4x-5}$

$\frac{2x-3}{4x-51}-\frac{6x-2}{4x-5}=\frac{(2x-3)-(6x-2)}{4x-5}$

keep the common denominator and subtract the numerators

$=\frac{2x-3-6x+2}{4x-5}$ *be sure to distribute the negative*

$=\frac{-4x-1}{4x-5}$ *combine like terms*

Example 2

Combine these fractions, simplifying your answer.

a $\frac{3x}{3x-1}+\frac{3x+1}{2x+5}$

$\frac{3x}{3x-1}+\frac{3x+1}{2x+5}=\frac{3x}{3x-1}\cdot\frac{2x+5}{2x+5}+\frac{3x+1}{2x+1}\times\frac{3x-1}{3x-1}$

multiply each fraction by one to get a common denominator

$=\frac{3x(2x+5)}{(3x-1)(2x+5)}+\frac{(3x+1)(3x-1)}{(2x+5)(3x-1)}$

$=\frac{(6x^2+15x)}{(3x-1)(2x+5)}+\frac{(9x^2-1)}{(2x+5)(3x-1)}$ *expand the brackets*

$=\frac{15x^2+15x-1}{(3x-1)(2x+5)}$ *combine like terms*

b $\dfrac{5x}{x+3}-\dfrac{2x+1}{2x-1}$

$\dfrac{5x}{x+3}-\dfrac{2x+1}{2x-1}=\dfrac{5x}{x+3}\cdot\dfrac{2x-1}{2x-1}-\dfrac{2x+1}{2x-1}\times\dfrac{x+3}{x+3}$

multiply each fraction by one to get a common denominator

$=\dfrac{5x(2x-1)}{(x+3)(2x-1)}-\dfrac{(2x+1)(x+3)}{(2x-1)(x+3)}$

$=\dfrac{(10x^2-5x)}{(x+3)(2x-1)}-\dfrac{(2x^2+7x+3)}{(2x-1)(x+3)}$ *expand the brackets*

$=\dfrac{10x^2-5x-2x^2-7x-3}{(x+3)(2x-1)}$ *watch out for negative signs*

$=\dfrac{8x^2-12x-3}{(x+3)(2x-1)}$ *combine like terms*

Exercise

1 Combine these fractions, simplifying your answer.

a $\dfrac{2}{x+7}+\dfrac{3x-1}{x+7}$ **b** $\dfrac{4x}{2x+2}-\dfrac{3x-1}{2x+2}$

c $\dfrac{3x+9}{3x+4}+\dfrac{3x-1}{3x+4}$ **d** $\dfrac{2x}{x+5}+\dfrac{x+1}{2x-1}$

e $\dfrac{4}{x}+\dfrac{2x+1}{x+2}$ **f** $\dfrac{2x-1}{x-2}-\dfrac{3x}{4x+3}$

g $\dfrac{x+1}{5x+1}+\dfrac{2x}{2x-5}$ **h** $\dfrac{x+5}{x-4}-\dfrac{x-2}{x+2}$

Answers

1 **a** $\frac{3x+1}{x+7}$ **b** $\frac{x+1}{2x+2}$ or $\frac{1}{2}$ **c** $\frac{6x+8}{3x+4}$ or 2 **d** $\frac{5x^2+4x+5}{(x+5)(2x-1)}$

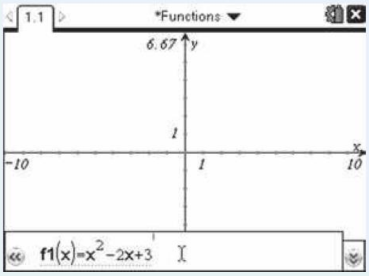
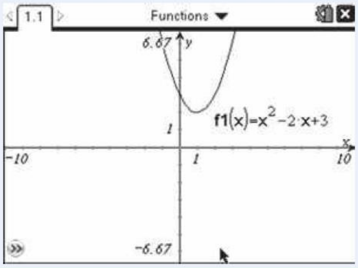
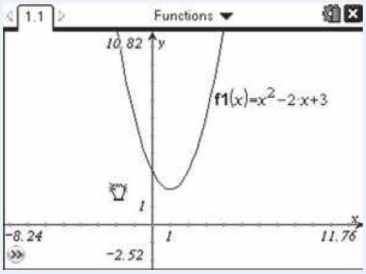
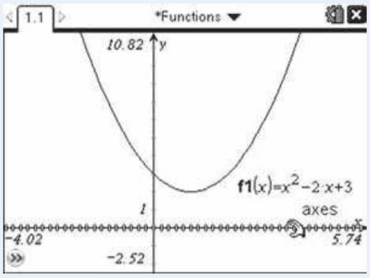
e $\frac{2x^2+5x+5}{x(x+2)}$ **f** $\frac{5x^2+8x-3}{(x-2)(4x+3)}$ **g** $\frac{12x^2-x-5}{(5x+1)(2x-5)}$ **h** $\frac{13x+2}{(x-4)(x+2)}$

Drawing a quadratic graph

Example

Draw the graph of $y = x^2 - 2x + 3$ and display using suitable axes.

Answer

Open a new document and add a Graphs page. The entry line is displayed at the bottom of the work area. The default graph type is Function, so the form '*f1(x)* =' is displayed. The default axes are $-10 \le x \le 10$ and $-6.67 \le y \le 6.67$.	
Type $x^2 - 2x + 3$ and press . The calculator displays the curve with the default axes.	
Pan the axes to get a better view of the curve. For help with panning, see your GDC manual.	
Grab the *x*-axis and change it to make the quadratic curve fit the screen better. For help with changing axes, see your GDC manual.	

Changing a quadratic from standard to vertex form

You can use completing the square to change a quadratic function from standard form to vertex form.

Example 1

Write $y = x^2 + 4x - 5$ in vertex form.

Answer

$y = x^2 + bx + c$

$\left(\frac{b}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 2^2 = 4$ *use* $c = \left(\frac{b}{2}\right)^2$ *to complete the square for the* x^2 *and* x *terms*

$x^2 + 4x + 4 = (x+2)^2$

$y = x^2 + 4x - 5 = (x^2 + 4x + 4) - 5 - 4$

write the complete square on the right -hand side; as this adds 4, subtract 4 at the end

$y = x^2 + 4x - 5 = (x+2)^2 - 9$

$y = (x+2)^2 - 9$

$(x+2)^2 - 9 = x^2 + 4x + 4 - 9 = x^2 + 4x - 5$

expand and simplify to check your answer

Generally, using the $y = ax^2 + bx + c$ form of the equation, $h = \frac{-b}{2a}$ and $k = f(h)$.

For $y = a(x-h)^2 + k$ the vertex is (h, k).

Exercise

Write each quadratic function in its vertex form.

1 $y = x^2 + 4x - 3$

2 $y = x^2 + 3x - 2$

3 $y = x^2 - 2x + 3$

4 $y = 3x^2 + 6x + 2$

5 $y = -x^2 - 6x + 1$

6 $y = 1 - 4x - x^2$

Answers

1 $y = (x+2)^2 - 7$

2 $y = (x+\frac{3}{2})^2 - \frac{17}{4}$

3 $y = 2(x-\frac{1}{2})^2 + \frac{5}{2}$

4 $y = 3(x+1)^2 - 1$

5 $y = 10 - (x+3)^2$

6 $y = 5 - (x+2)^2$

Solving quadratic equations by factorizing

Consider the quadratic equation $x^2 + 2x - 8 = 0$.

Factor the left side into $(x + 4)(x - 2) = 0$.

Since we have a product that is equal to zero, one of the factors must equal o, hence either $x + 4$ = o and $x = -4$, or $x - 2$ = o and $x = 2$.

There are two unique solutions, $x = -4$ or $x = 2$.

Example 1

Solve the quadratic equation $2x^2 + 5x - 3 = 0$ by factorizing.

Answer

$2x^2 + 5x - 3 = (2x - 1)(x + 3) = 0$	*factorize*
$2x - 1 = 0$ or $x + 3 = 0$	*set each factor equal to 0*
$x = \frac{1}{2}$ or $x = -3$	*solve each linear equation*

Example 2

Solve the quadratic equation $4x - x^2 = 4$ by factorizing.

Answer

$4x - x^2 = 4$ so $-x^2 + 4x - 4 = 0$	*set the equation equal to 0*
$x^2 - 4x + 4 = 0$	*multiply both sides by -1*
$x^2 - 4x + 4 = (x - 2)^2 = 0$	*factorize*
$x - 2 = 0$	*set the linear factor equal to 0*
$x = 2$	*solve*

In this case there is only one unique solution, or a repeated solution.

Exercise

Solve these quadratic equations by first factorizing.

1 $x^2 - 8x + 15 = 0$

2 $x^2 + 6x - 16 = 0$

3 $x^2 - 8x + 16 = 0$

4 $28 + 3x = x^2$

5 $6x^2 + 7x - 3 = 0$

6 $-2x^2 = 3x - 2$

Answers

1 $x = 3$ or $x = 5$

2 $x = -8$ or $x = 2$

3 $x = 4$

4 $x = -4$ or $x = 7$

5 $x = -\frac{3}{2}$ or $x = \frac{1}{3}$

6 $x = -2$ or $x = \frac{1}{2}$

Solving quadratic equations by completing the square

Example 1

Solve the quadratic equation $x^2 + 4x - 5 = 0$ by completing the square.

Answer

$x^2 + 4x - 5 = 0$ so $x^2 + 4x = 5$ *bring the constant to the right-hand side*

$x^2 + 4x + 4 = 5 + 4$

complete the square on the left-hand side and add the constant to the right-hand side

$(x+2)^2 = 9$ *write the trinomial as a square of a binomial*

$x + 2 = \pm 3$ *take the square root of both sides*

$x = -2 \pm 3$ *solve for x*

$x = -2 + 3 = 1$ or $x = -2 - 3 = -5$ *separate both solutions*

Example 2

Solve the quadratic equation $2x^2 - 4x = 3$ by completing the square.

Answer

$2x^2 - 4x = 2(x^2 + 2) = 3$ *take out a factor of 2 from the first two terms*

$x^2 - 2 = \frac{3}{2}$ *divide both sides by 2*

$x^2 - 2x + 1 = \frac{3}{2} + 1$

complete the square on the left-hand side and add the constant to the right-hand side

$(x-1)^2 = \frac{5}{2}$ *write the trinomial as a square of a binomial*

$x - 1 = \pm\sqrt{\frac{5}{2}} = \pm\frac{\sqrt{10}}{2}$ *take the square root of both sides*

$x = 1 \pm \frac{\sqrt{10}}{2}$ *solve for x*

$x = 1 + \frac{\sqrt{10}}{2}$ or $x = 1 - \frac{\sqrt{10}}{2}$ *separate both solutions*

Example 3

Solve the quadratic equation $x^2 + 2x = -4$ by completing the square.

Answer

$x^2 + 2x + 1 = -4 + 1$

complete the square on the left-hand side and add the constant to the right-hand side

$(x+1)^2 = -3$ *you cannot take the square root of the right-hand side*

No solution

Some quadratic equations have no real solutions.

Exercise

Solve these quadratic equations by completing the square.

1 $x^2 + 4x = 3$

2 $x^2 + 3x = 2$

3 $2x^2 - 2x = 1$

4 $3x^2 + 6x = -2$

Answers

1 $x = -2 - \sqrt{7}$ or $x = \sqrt{7} - 2$

2 $x = -\frac{3}{2} - \frac{\sqrt{17}}{2}$ or $x = \frac{\sqrt{17}}{2} - \frac{3}{2}$

3 $x = \frac{1}{2} - \frac{\sqrt{3}}{2}$ or $x = \frac{1}{2} + \frac{\sqrt{3}}{2}$

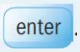
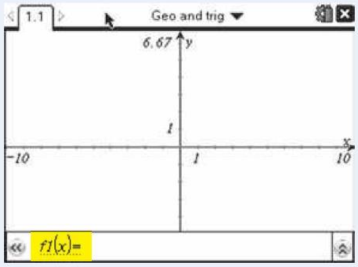
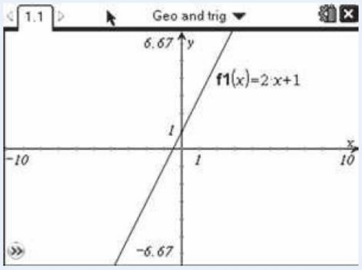
4 $x = -1 - \frac{1}{\sqrt{3}}$ or $x = \frac{1}{\sqrt{3}} - 1$

Graphing linear functions

Example

Draw the graph of the function $y = 2x + 1$

Answer

Open a new document and add a Graphs page. The entry line is displayed at the bottom of the work area. The default graph type is Function, so the form '*f1(x)* =' is displayed. The default axes are $-10 \le x \le 10$ and $-6.67 \le y \le 6.67$. Type $2x + 1$ and press  .	
The graph of $y = 2x + 1$ is now displayed and labelled on the screen.	

Drawing a quadratic graph

Example

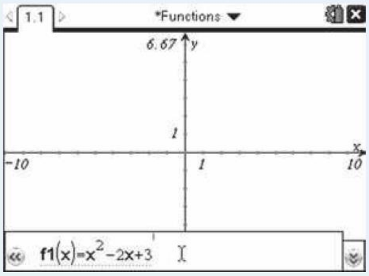
Draw the graph of $y = x^2 - 2x + 3$ and display using suitable axes.

Answer

Open a new document and add a Graphs page.

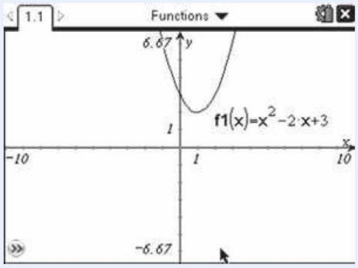
The entry line is displayed at the bottom of the work area. The default graph type is Function, so the form '*f1(x)* =' is displayed.

The default axes are $-10 \le x \le 10$ and $-6.67 \le y \le 6.67$.



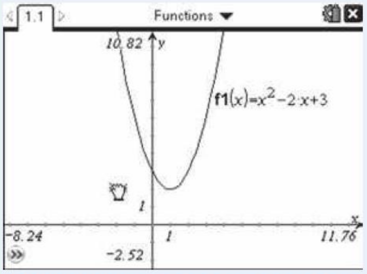
Type $x^2 - 2x + 3$ and press enter.

The calculator displays the curve with the default axes.



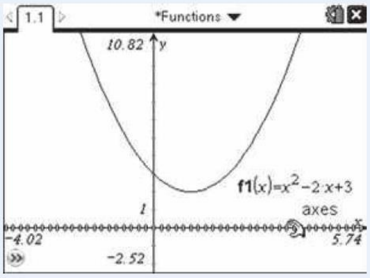
Pan the axes to get a better view of the curve.

For help with panning, see your GDC manual.



Grab the x-axis and change it to make the quadratic curve fit the screen better.

For help with changing axes, see your GDC manual.



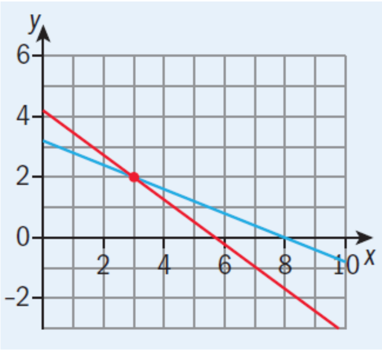
Simultaneous linear equations

Simultaneous equations involve two (or more) variables. There are two methods which you can use, called substitution and elimination.

Example 1

Solve the equations $3x + 4y = 17$ and $2x + 5y = 16$.

Answer



Geometrically you could consider these two linear equations as the equations of two straight lines.

Finding the solution to the equation is equivalent to finding the point of intersection of the lines.

The coordinates of the point will give you the values for x and y.

Substitution method

$3x + 4y = 17$ — *rearrange one of the equations to make y the subject*

$2x + 5y = 16$

$5y = 16 - 2x$

$y = \frac{16}{5} - \frac{2}{5}x$

$3x + 4(\frac{16}{5} - \frac{2}{5}x) = 17$ — *substitute for y in the other equation*

$3x + \frac{64}{5} - \frac{8}{5}x = 17$

$15x + 64 - 8x = 85$

$15x - 8x = 85 - 64$

$7x = 21$ — *solve the equation for x*

$x = 3$

$3(3) + 4y = 17$ — *substitute for x in one of the original equations*

$9 + 4y = 17$ — *solve for y*

$4y = 8$

$y = 2$

The solution is x=3, y=2.

Elimination method

$3x + 4y = 17 \rightarrow (1)$
$2x + 5y = 16 \rightarrow (2)$

multiply equation (1) by 2 and equation (2) by 3 to make the coefficients of x equal

$6x + 8y = 34 \rightarrow (3)$
$6x + 15y = 48 \rightarrow (4)$

Subtract the equations [(4)-(3)] to eliminate x from the equations

$7y = 14$

$y = 2$

$3x + 4(2) = 17$

$3x + 8 = 17$

$3x = 9$

$x = 3$

The solution is x=3, y=2.

Exercise

1 Solve these simultaneous equations using substitution.

a $y = 3x - 2$ and $2x + 3y = 5$

b $4x - 3y = 10$ and $2y + 5 = x$

c $2x + 5y = 14$ and $3x + 4y = 7$

2 Solve these simultaneous equations using elimination.

a $2x - 3y = 15$ and $2x + 5y = 7$

b $3x + y = 5$ and $4x - y = 9$

c $x + 4y = 6$ and $3x + 2y = -2$

d $3x + 2y = 8$ and $2x + 3y = 7$

e $4x - 5y = 17$ and $3x + 2y = 7$

Answers

1 **a** $x = 1, y = 1$ **b** $x = 1, y = -2$ **c** $x = -3, y = 4$

2 **a** $x = 6, y = -1$ **b** $x = 2, y = -1$ **c** $x = -2, y = 2$ **d** $x = 2, y = 1$
e $x = 3, y = -1$

Measures of central tendency

A measure of central tendency, or average, describes a typical value for a set of data.

There are three common types of average:

- The mode - this is the data value that occurs most often.
- The median - this is the middle item when the data is arranged in order of size.
- The mean - this is what most people mean when they use the word "average". It is found by adding up all of the data and dividing by the number of pieces of data.

Example 1

Find **a** the mode **b** the median and **c** the mean of this data set:

2, 5, 4, 9, 1, 3, 2, 6, 9, 2, 5, 1 3, 4

Answers

a The mode is 2 — *2 occurs the most often*

b 1, 2, 2, 2, 3, 4, 4, 5, 5, 6, 9, 9, 1 3 — *Write them in order and find the middle one*

The median is 4

c Mean $= \dfrac{1+2+2+2+3+4+4+5+5+6+9+9+13}{13}$ — *Add them all together. There*

$= \dfrac{65}{13} = 5$ — *are 13 pieces of data, so*

divide by 13.

Exercise

1 Find **a** the mode **b** the median and **c** the mean of

a 1, 4, 1 , 5, 6, 7, 3, 1 , 8

b 4, 7, 5, 12, 5, -3, -2

c 2, 3, 8, 2, 1, 7, 9, 8, 5

d 25, 28, 29, 21, 25, 20, 27

e 7.4, 10.2, 12.5, 6.8, 10.2

2 Fifteen students were asked how many brothers and sisters they had. The results were:

2, 2, 1, 0, 3, 5, 2, 1 ,1, 0, 1, 4, 1, 0, 2.

Find **a** the mode, **b** the median and **c** the mean number of brothers and sisters.

3 My last nine homework scores, marked out of 10, were:

8, 7, 9, 10, 8, 9, 6, 8, 7

Find **a** the mode **b** the median and **c** the mean homework score.

4 A sprinter's times in seconds for the 40 m dash were:

5.1 3, 4.82, 5.25, 4.94, 5.06, 4.82, 5.12

Find **a** the mode, **b** the median and **c** the mean of the times.

5 Seven farmers own different numbers of chickens.

These numbers are:

253, 78, 497, 166, 710, 497 and 599

Find **a** the mode, **b** the median and **c** the mean number of chickens.

Answers

1	**a** mode=1, median=4, mean=4		**b** mode=5, median=5, mean=4
	c mode=2 and 8, median=5, mean=5		**d** mode=25, median=25, mean=25
	e mode=10.2, median=10.2, mean=9.42		
2	**a** 1	**b** 1	**c** 1 .67
3	**a** 8	**b** 8	**c** 9
4	**a** 4.82	**b** 5.06	**c** 5.02
5	**a** 497	**b** 497	**c** 400

Measures of dispersion

A measure of dispersion is a value that describes the spread of a set of data.

The **range** and **interquartile range** are two measures of dispersion. The range shows how spread out the data is.

Range = highest value - lowest value

The quartiles divide a set of data into four equal amounts.

The **lower quartile** Q_1 is 25% of the way through the data and its position is found using the formula:

$Q_1 = \frac{n+1}{4}$, where n is the number of items in the data set.

The **upper quartile** Q_3 is 75% of the way through the data and its position is found using the formula:

$Q_3 = 3\left(\frac{n+1}{4}\right)^{th}$ value

The **interquartile range** shows how spread out the middle 50% of the data is.

Inter quartile range = Q_3 - Q_1

Example 1

Here are the shoe sizes of fifteen boys:

42, 42, 38, 40, 42, 40, 34, 46, 44, 36, 38, 40, 42, 36, 42

Find

a the range **b** the interquartile range.

Answers

a 34, 36, 36, 38, 38, 40, 40, 40, 42, 42, 42, 42, 42, 44, 46 *arrange the data in order of size*

range = 46 - 34 = 12

b Lower quartile $= \frac{16}{4}$ th value *n=15*

=4th value

=38

Upper quartile = 3 × 4th value

=12th value

=42

Interquartile range = 42 - 38 = 4

Exercise

1 Here are the shoe sizes of fifteen girls:

26, 28, 28, 36, 34, 32, 30, 34, 32, 28, 36, 38, 34, 32, 30

Find **a** the range and **b** the interquartile range of the shoe sizes.

2 23 students were asked how many pets they had at home.

Here are the replies:

1, 4, 3, 5, 3, 2, 8, 0, 2, 1, 3, 2, 4, 2, 1, 0, 1, 2, 6, 7, 2, 8, 2

Find **a** the range and **b** the interquartile range for the number of pets.

3 The average daily temperatures in °C in Bucharest during the 31 days of January were:

-6, -4, -4, -2, -1, 0, 4, 5, 7, 4, 2, 1, 0, -3, -4, -6, -7, -5, -3, -1, 1, 3, 4, 7, 7, 8, 3, -2, 0, -2, -5

Find **a** the range and **b** the interquartile range for the daily temperatures.

4 The grocer sells potatoes by the kilogram. I bought 1kg of potatoes every day of the week and counted the number of potatoes each time. Here are the results:

Day	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Potatoes	18	15	20	17	14	12	15

Find **a** the range and **b** the interquartile range for the number of potatoes in 1 kilogram.

5 The time (in seconds) taken for eleven players in a soccer team to prepare for a free kick is given:

12.4, 2.45, 3.75, 10, 3.5, 8.4, 9.6, 23.5, 2.48, 15.6, 5.2

Find **a** the range and **b** the interquartile range for the time taken.

Answers

1 **a** 38 – 26 = 12 **b** 34 – 28 = 6

2 **a** 8 – 0 = 8 **b** 4 – 1 = 3

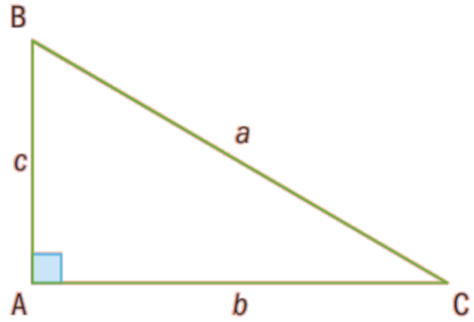
3 **a** 8 – (-7) = 15 **b** 4 – (-4) = 8

4 **a** 20 – 12 = 8 **b** 18 – 14 = 4

5 **a** 23.5 – 2.45 = 21.05 **b** 12.4 – 3.5 = 8.9

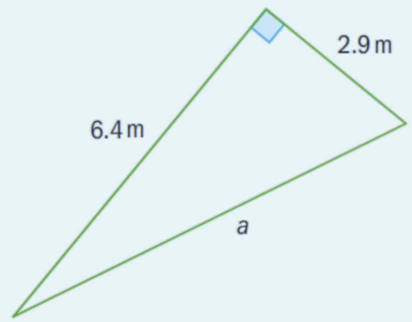
Pythagoras' theorem

In a right-angled triangle *ABC* with sides *a*, *b* and *c*, where *a* is the hypotenuse: $a^2 = b^2 + c^2$



Example 1

Find the length marked *a*.



Answer

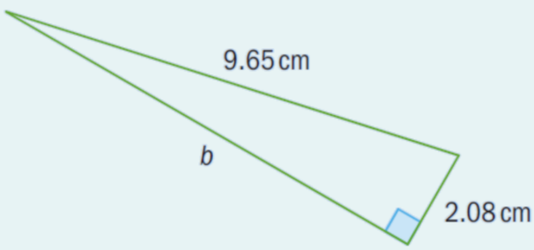
$a^2 = 6.4^2 + 2.9^2$

$a = \sqrt{6.4^2 + 2.9^2} = 7.03$ cm

You can use Pythagoras' Theorem to calculate the length of one side of a right-angled triangle when you know the other two.

Example 2

Find the length marked *b*.



Answer

$9.65^2 = b^2 + 2.08^2$

$b^2 = 9.65^2 - 2.08^2$

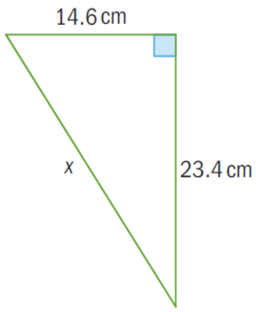
$a = \sqrt{9.65^2 - 2.08^2} = 9.42$ cm

Check your answer by making sure that the hypotenuse is the longest side of the triangle.

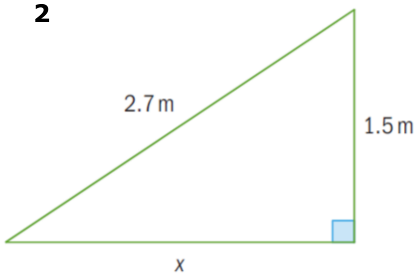
Exercise

In each diagram, find the length of the side marked x. Give your answer to 3 significant figures.

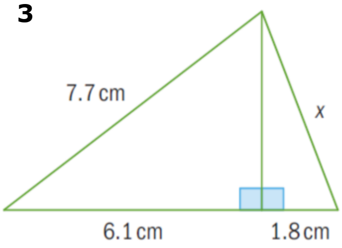
1



2



3



Answers

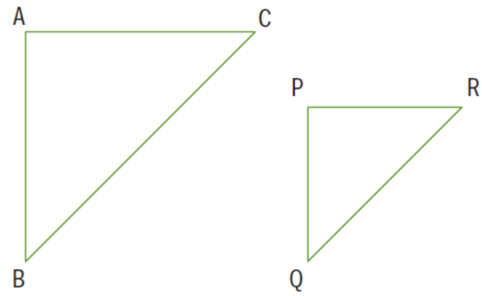
1 27.6 cm

2 2.24 cm

3 5.03 cm

Similar triangles

In similar triangles, corresponding angles are equal and corresponding sides are in the same ratio.



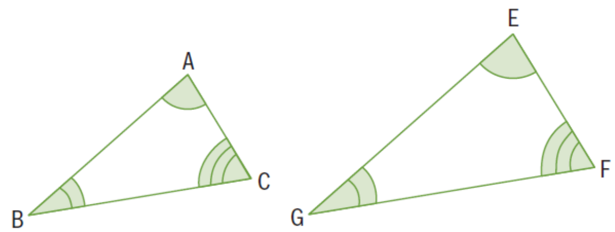
Triangles ABC and PQR are similar because

$\hat{A}=\hat{P}, \hat{B}=\hat{Q}, \hat{C}=\hat{R}$

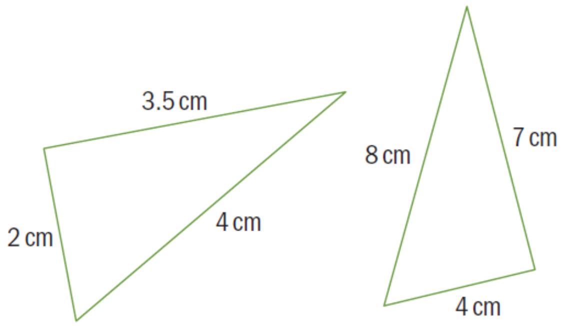
$\frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR}=\text{scale factor}$

To prove that two triangles are similar, show that **one** of these three statements is true:

1 The three angles of one triangle are equal to the three angles of the other triangle

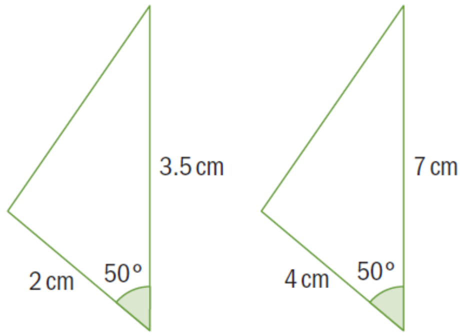


2 The corresponding sides of each triangle are in the same ratio



$$\frac{8}{4}=\frac{4}{2}=\frac{7}{3.5}=2$$

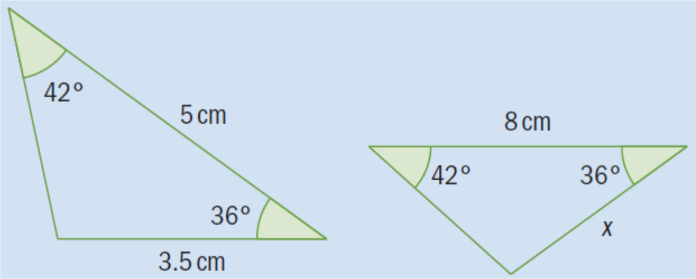
3 There is one pair of equal angles and the sides containing these angles are in the same ratio



$$\frac{4}{2} = \frac{7}{3.5} = 2$$

Example

Find the length of the side marked *x*.



Answer

Two pairs of angles are equal, so the third pair must be equal. *prove similarity*

Hence the triangles are similar.

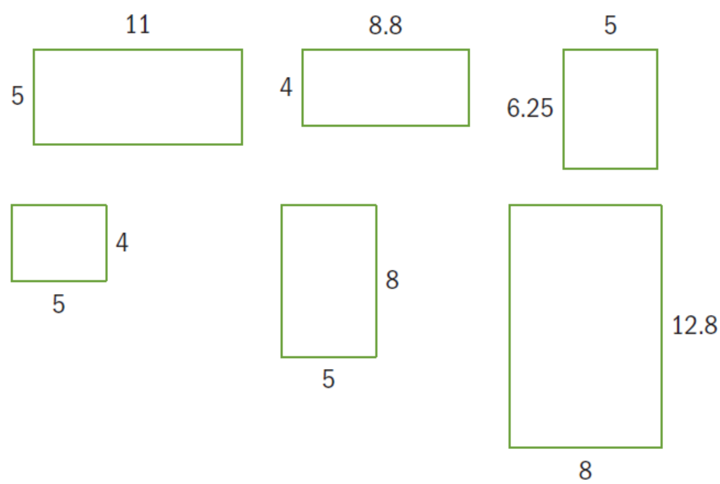
The scale factor of the enlargement is $\frac{8}{5} = 1.6$

So $x = 3.5 \times 1.6 = 5.6$ cm

Exercise

Note that the shapes in this exercise are not drawn to scale.

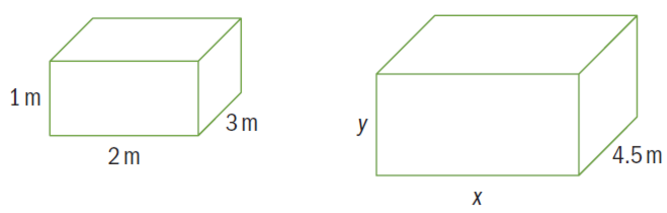
1 Which pairs of rectangles are similar?



2 These shapes are similar. Calculate the lengths marked by letters.

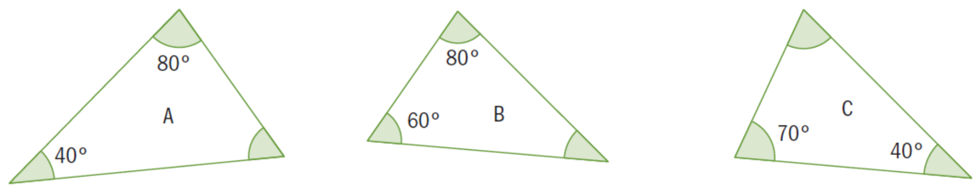
a

b

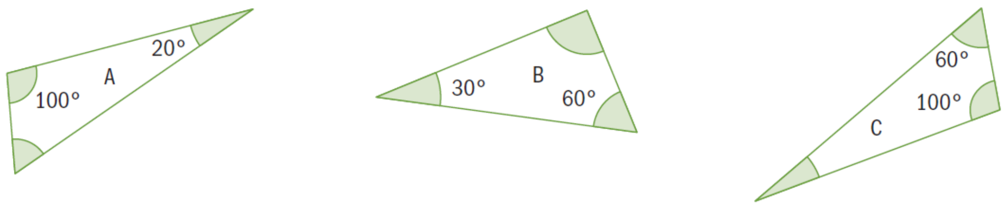


3 Which triangles are similar?

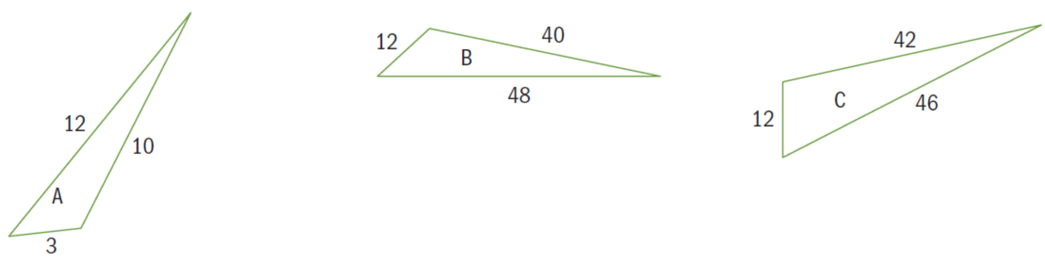
a



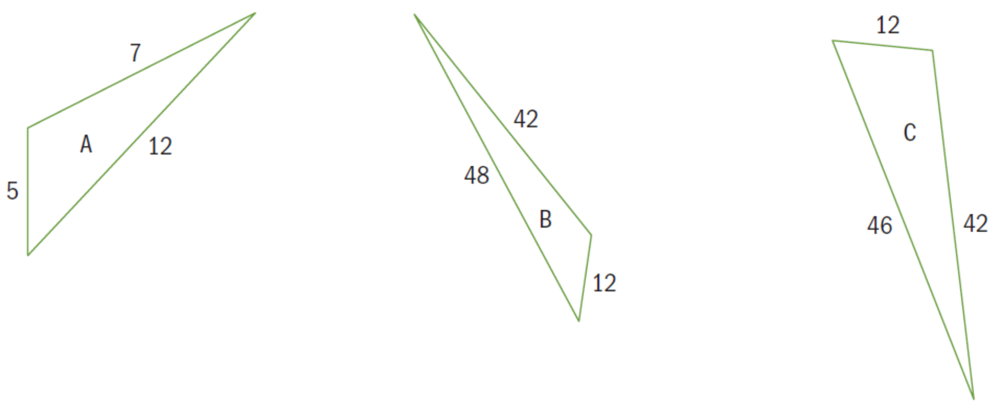
b



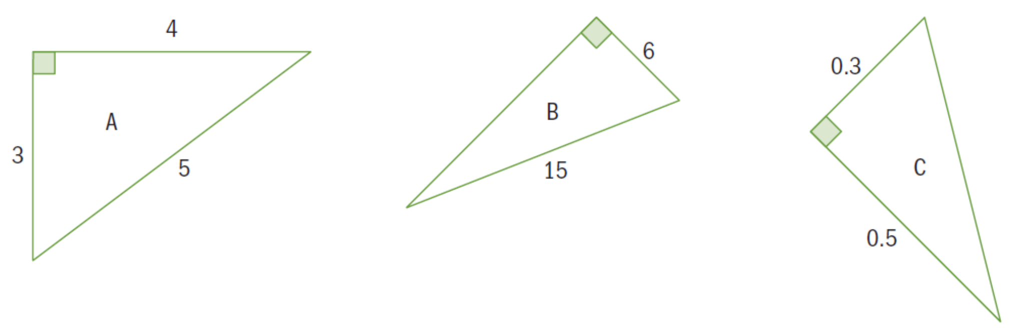
c



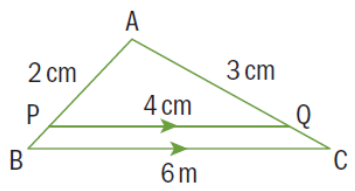
d



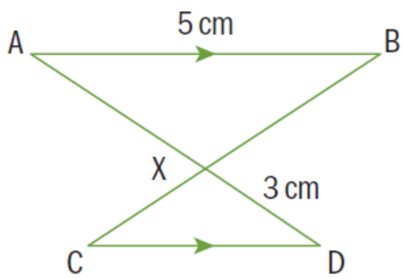
e



4 Show that triangles ABC and APQ are similar. Calculate the length of AC and BP.



5 In the diagram, AB and CD are parallel, and AD and BC meet at X.



a Prove that triangles ABX and DCX are similar.

b Which side in triangle DCX corresponds to AX in triangle ABX?

c If CD=3.8 cm, what is the length of AX?

Answers

1 Rectangles with sides 5, 11 and 4, 8.8

Rectangles with sides 5, 6.25 and 4, 5

Rectangles with sides 5, 8 and 8, 12.8

2 **a** Scale factor is $10.08 \div 7.2 = \frac{7}{5}$; $y = 9.1 \times \frac{5}{7} = 6.5$ cm; $x = 13 \times \frac{7}{5} = 18.2$ cm

b Scale factor is $4.5 \div 3 = 1.5$; $y = 1 \times 1.5 = 1.5$ cm; $x = 2 \times 1.5 = 3$ cm

3 **a** A and B **b** A and C **c** A and B

d None **e** None

4 Angle PAQ = Angle BAC

Angle ABC = Angle APQ (parallel lines and corresponding angles)

Angle ACB = Angle AQP (parallel lines and corresponding angles)

Hence similar triangles

Scale factor is $\frac{6}{4}$ or 1.5

So $AB = 2 \times 1.5 = 3$ cm and $BP = AB - AP = 3 - 2 = 1$ cm

So $AC = 3 \times 1.5 = 4.5$ cm

5 **a** Angle AXB = Angle CXD (vertically opposite angles)

Angle BAX = Angle XDC (parallel lines and alternate angles)

Angle ABX = Angle XCD (parallel lines and alternate angles)

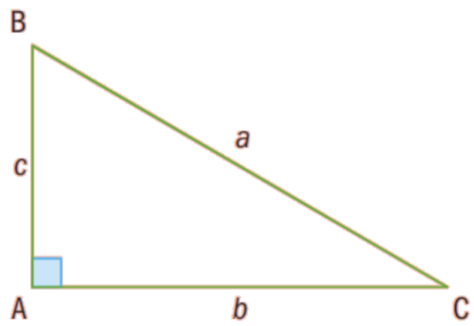
Hence similar triangles

b XD

c 3.9 cm

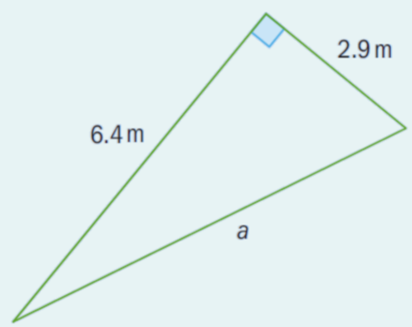
Pythagoras' theorem

In a right-angled triangle *ABC* with sides *a*, *b* and *c*, where *a* is the hypotenuse: $a^2 = b^2 + c^2$



Example 1

Find the length marked *a*.



Answer

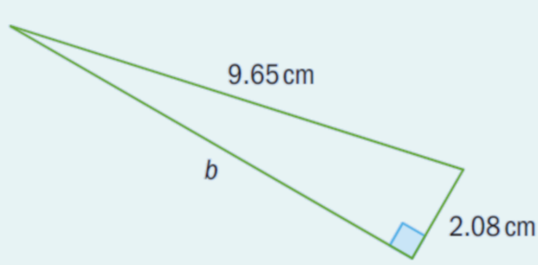
$a^2 = 6.4^2 + 2.9^2$

$a = \sqrt{6.4^2 + 2.9^2} = 7.03$ cm

You can use Pythagoras' Theorem to calculate the length of one side of a right-angled triangle when you know the other two.

Example 2

Find the length marked *b*.



Answer

$9.65^2 = b^2 + 2.08^2$

$b^2 = 9.65^2 - 2.08^2$

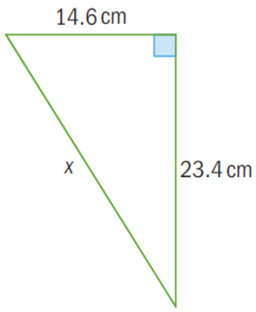
$a = \sqrt{9.65^2 - 2.08^2} = 9.42$ cm

Check your answer by making sure that the hypotenuse is the longest side of the triangle.

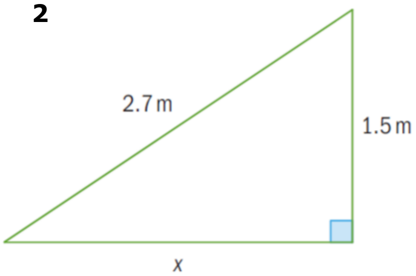
Exercise

In each diagram, find the length of the side marked x. Give your answer to 3 significant figures.

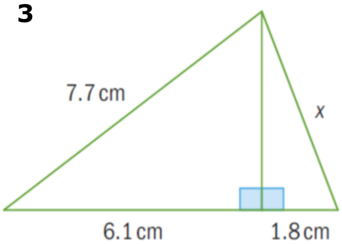
1



2



3



Answers

1 27.6 cm

2 2.24 cm

3 5.03 cm

Coordinates

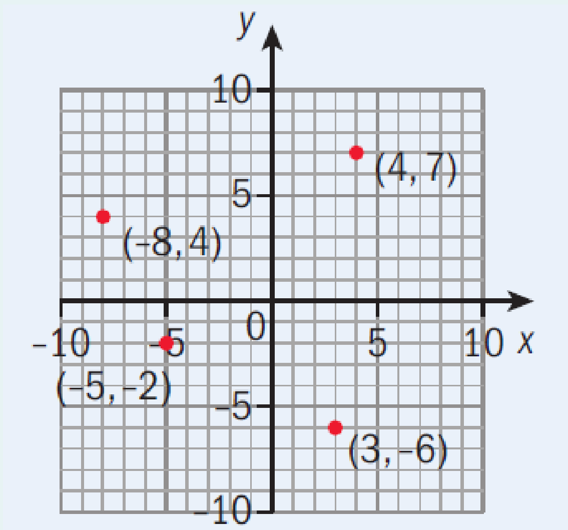
Coordinates describe the position of points in the plane. Horizontal positions are shown on the *x*-axis and vertical positions on the *y*-axis.

Example

Draw axes for and -10 ≤ x ≤ 10 and -10 ≤ y ≤ 10.

Plot the points with coordinates: (4, 7), (3, -6), (-5, -2) and (-8, 4).

Answer

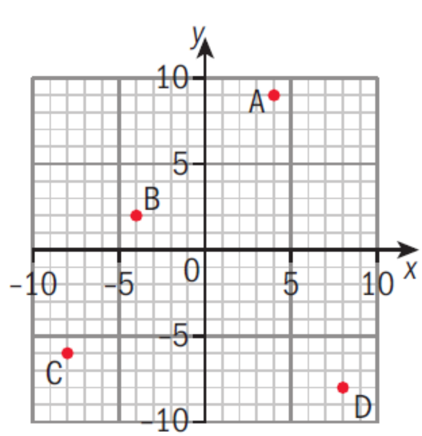


Exercise

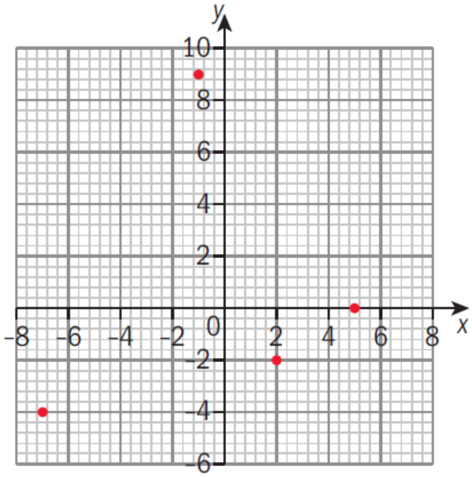
1 Draw axes for -8 ≤ x ≤ 8 and -5 ≤ y ≤ 10.

Plot the points with coordinates: (5, 0), (2, -2), (-7, -4) and (-1, 9).

2 Write down the coordinates of the points shown in this diagram.



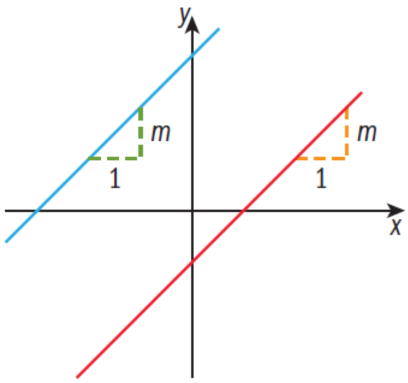
Answers

1 

2 A(4, 9), B(-4, 2), C(-8, -6), D(8, -8)

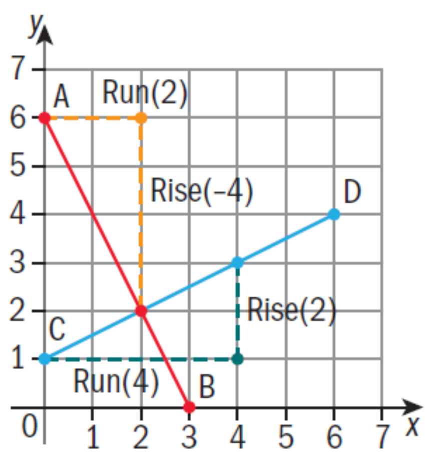
Parallel and perpendicular lines

Parallel lines have the same gradient.



Both of these lines have gradient m.

Perpendicular lines have gradients m and $-\frac{1}{m}$.



Line CD has gradient $\frac{1}{2}$. Line AB had gradient -2.

Notice that the product of perpendicular gradients is -1:

$$-2 \times \frac{1}{2} = -1$$

Exercise

1 **a** Which of these gradients are parallel?

b Which are perpendicular?

$$3, -3, \frac{1}{3}, 4.5, \frac{2}{3}, \frac{2}{9}, -\frac{2}{9}, -1.5, \frac{6}{2}$$

2 State if the lines in each pair are parallel, perpendicular or neither.

a Line A through (2, 5) and (0, 1) and line B though (4, 10) and (5, 12).

b Line C through (3, 14) and (-2, -6) and line D though (12, -3) and (20, -5).

c Line E through (1, 10) and (5, 15) and line F through (2, 2) and (4, 2).

d Line G through (5, 7) and (2, 4) and line H through (8, -5) and (4, -1).

e Line I through (4, 11) and (10, 20) and line J through (2, 1).

Answers

1 **a** 3 and $\frac{6}{2}$, 4.5 and $\frac{9}{2}$

b -3 and $\frac{1}{3}$, 4.5 and $-\frac{2}{9}$, $\frac{2}{3}$ and -1.5

2 **a** Parallel (both have a gradient of 2)

b Perpendicular (one has a gradient of 4 and the other's is $-\frac{1}{4}$)

c Neither (one has a gradient of $\frac{5}{4}$ and the other's is 0)

d Perpendicular (gradients of 1 and -1)

e Parallel (both have a gradient of 1.5)

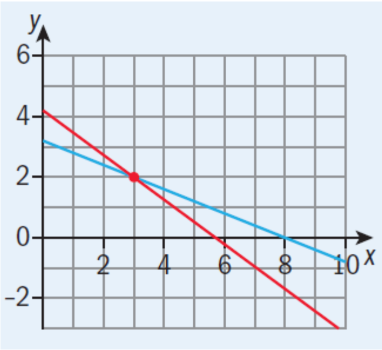
Simultaneous linear equations

Simultaneous equations involve two (or more) variables. There are two methods which you can use, called substitution and elimination.

Example 1

Solve the equations $3x + 4y = 17$ and $2x + 5y = 16$.

Answer



Geometrically you could consider these two linear equations as the equations of two straight lines.

Finding the solution to the equation is equivalent to finding the point of intersection of the lines.

The coordinates of the point will give you the values for x and y.

Substitution method

$3x + 4y = 17$ *rearrange one of the equations to make y the subject*

$2x + 5y = 16$

$5y = 16 - 2x$

$y = \frac{16}{5} - \frac{2}{5}x$

$3x + 4(\frac{16}{5} - \frac{2}{5}x) = 17$ *substitute for y in the other equation*

$3x + \frac{64}{5} - \frac{8}{5}x = 17$

$15x + 64 - 8x = 85$

$15x - 8x = 85 - 64$

$7x = 21$ *solve the equation for x*

$x = 3$

$3(3) + 4y = 17$ *substitute for x in one of the original equations*

$9 + 4y = 17$ *solve for y*

$4y = 8$

$y = 2$

The solution is $x=3$, $y=2$.

Elimination method

$3x + 4y = 17 \rightarrow (1)$
$2x + 5y = 16 \rightarrow (2)$

multiply equation (1) by 2 and equation (2) by 3 to make the coefficients of x equal

$6x + 8y = 34 \rightarrow (3)$
$6x + 15y = 48 \rightarrow (4)$

Subtract the equations [(4)-(3)] to eliminate x from the equations

$7y = 14$

$y = 2$

$3x + 4(2) = 17$

$3x + 8 = 17$

$3x = 9$

$x = 3$

The solution is x=3, y=2.

Exercise

1 Solve these simultaneous equations using substitution.

a $y = 3x - 2$ and $2x + 3y = 5$

b $4x - 3y = 10$ and $2y + 5 = x$

c $2x + 5y = 14$ and $3x + 4y = 7$

2 Solve these simultaneous equations using elimination.

a $2x - 3y = 15$ and $2x + 5y = 7$

b $3x + y = 5$ and $4x - y = 9$

c $x + 4y = 6$ and $3x + 2y = -2$

d $3x + 2y = 8$ and $2x + 3y = 7$

e $4x - 5y = 17$ and $3x + 2y = 7$

Answers

1 **a** $x = 1, y = 1$ **b** $x = 1, y = -2$ **c** $x = -3, y = 4$

2 **a** $x = 6, y = -1$ **b** $x = 2, y = -1$ **c** $x = -2, y = 2$ **d** $x = 2, y = 1$
e $x = 3, y = -1$